Solving equilibrium problems using extended mathematical programming

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What we can do?

- Equilibrium $\equiv$ complementarity ($\approx$ coupling)
- PATH solver for large scale mixed complementarity problems

$$0 \leq F(x) \perp x \geq 0$$

- Nonsmooth Newton method, efficient linear algebra, available in modeling systems: GAMS, MPSGE, AMPL, AIMMS, Julia, Pyomo
- Used in models such as PIES, MERGE, VEMOD, MARKAL, TIMES, KAPSARC, ISEEM, MESSAGE, TEA, TIGER, Gemstone
- Models of Tobin, Nordhaus, Romer
- Frequently used in Computable General Equilibrium (CGE) analyses (GTAP data available), traffic, structural analysis
- Policy analyses such as Uruguay round, NAFTA, USMCA, Brexit

Equilibrium = the first-order optimality conditions (KKTs)

An equilibrium of a single optimization (a single agent) under CQs

$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & f(x), & \nabla f(x) - \nabla g(x)^T \lambda - \nabla h(x)^T \mu = 0, \\
\text{subject to} & g(x) \leq 0, \;\; (\Rightarrow) & 0 \geq g(x) \perp \lambda \leq 0, \\
 & h(x) = 0, & 0 = h(x) \perp \mu,
\end{array}
$$

- Mixed complementarity problem $\text{MCP}([l, u], F) : l \leq z \leq u \;\perp\; F(z)$

Geometric first-order optimality conditions for a closed convex set K

$$
\underset{x \in K}{\text{minimize}} \quad f(x), \;\; (\Rightarrow) \quad 0 \in \nabla f(x) + N_K(x)
$$

$$
\text{i.e. } \mathrm{VI}(K, \nabla f(x))
$$

- Variational inequality $\mathrm{VI}(K, F) : \langle F(x), y - x \rangle \geq 0, \forall y \in K$

Generalizing to N agents: NEP

Nash equilibrium problem: $x = [x_i]_{i=1}^N$

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}), & \nabla_{x_i} f_i(x_i, x_{-i}) - \nabla g_i(x_i)\lambda_i - \nabla h_i(x_i)\mu_i = 0, \\ \text{subject to} & g_i(x_i) \leq 0, \ (\Rightarrow) & 0 \geq g_i(x_i) \perp \lambda_i \leq 0, \\ & h_i(x_i) = 0, & 0 = h_i(x_i) \perp \mu_i. \end{array}$$

- $x_{-i} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_N)^T$.
- Equilibrium: satisfy the KKT conditions of all agents simultaneously.
- Interactions occur only in objective functions.
- Example of an interaction: $f_i(x_i, x_{-i}) = c_i(x_i) - x_i p\left(\sum_{j=1}^N x_j\right)$

NEP + non-disjoint feasible regions: GNEP

Generalized Nash equilibrium problem: $x = [x_i]_{i=1}^N$

$$\begin{aligned} \underset{x_i}{\text{minimize}} \quad & f_i(x_i, x_{-i}), \\ \text{subject to} \quad & g_i(x_i, x_{-i}) \le 0, \ (\Rightarrow) \\ & h_i(x_i, x_{-i}) = 0, \end{aligned}$$

$$\begin{aligned} \nabla_{x_i} f_i(x) - \nabla_{x_i} g_i(x)\lambda_i - \nabla_{x_i} h_i(x)\mu_i = 0, \\ 0 \ge g_i(x) \perp \lambda_i \le 0, \\ 0 = h_i(x) \perp \mu_i. \end{aligned}$$

- Interactions occur in both objective functions and constraints.
- Non-disjoint feasible region:
  $K_i(x_{-i}) = \{x_i \in \mathbb{R}^{n_i} \mid g_i(x_i, x_{-i}) \le 0, h_i(x_i, x_{-i}) = 0\}$.
  - $K_i : \mathbb{R}^{n-n_i} \rightrightarrows \mathbb{R}^{n_i}$ a set-valued mapping
  - e.g., shared resources among agents: $\sum_{i=1}^N x_i \le b$, or strategic interactions
  - Quasi-variational inequality

(G)NEP + VI agent: MOPEC

Multiple optimization problems with equilibrium constraints:
$x = [x_i]_{i=1}^N, \pi$

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}, \pi), & \nabla_{x_i} f_i(x,\pi) - \nabla_{x_i} g_i(x,\pi)\lambda_i - \nabla_{x_i} h_i(x,\pi)\mu_i = 0, \\ \text{subject to} & g_i(x_i, x_{-i}, \pi) \le 0, & 0 \ge g_i(x,\pi) \perp \lambda_i \le 0, \\ & h_i(x_i, x_{-i}, \pi) = 0, & 0 = h_i(x,\pi) \perp \mu_i, \end{array}$$

$$\pi \quad \in \mathrm{SOL}(K, F), \qquad \pi \in K(x), \langle F(\pi, x), y - \pi \rangle \ge 0, \ \forall y \in K(x).$$

- No hierarchy between agents, c.f., MPECs and EPECs
- An example of a VI agent: market clearing conditions

$$0 \le \text{supply} - \text{demand} \quad \perp \quad \text{price} \ge 0$$

Specifying (G)NEPs and MOPECs in modeling languages

- Existing method
  1. Compute an MCP function F using the KKT conditions.

$$\begin{array}{lll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}), & \Longrightarrow \quad F_i(x, \lambda_i) = \begin{bmatrix} \nabla_{x_i} f_i - \nabla_{x_i} g_i \lambda_i \\ g_i \end{bmatrix}, \\ \text{subject to} & g_i(x_i, x_{-i}) \leq 0, & \\ & \text{for } i = 1, \ldots, N, & \quad \text{for } i = 1, \ldots, N. \end{array}$$

  2. Specify the complementarity relationship.

$$\begin{array}{ll} F \ \texttt{complements} \ (x, \lambda) & \text{in AMPL}, \\ F \ \perp \ (x, \lambda) & \text{in GAMS}. \end{array}$$

  3. Solve the resultant MCP$((x, \lambda), F)$ using the PATH solver.

  - Cons
    - Prone to errors as we require users to compute derivatives by hand
    - Not easy to modify the problem: a lot of derivative recomputations
    - Agent information is lost in the MCP function F.

The EMP framework

- Automates all the previous steps: no need to derive MCP by hand.
- Annotate equations and variables in an `empinfo` file.
- The framework automatically transforms the problem into another computationally more tractable form.

$$
\begin{aligned}
\underset{x_i}{\text{minimize}} \quad & f_i(x_i, x_{-i}, \pi), \\
\text{subject to} \quad & g_i(x_i, x_{-i}, \pi) \leq 0, \\
& h_i(x_i, x_{-i}, \pi) = 0, \\
& \text{for } i = 1, \dots, N, \\
\hline
\pi \quad & \in \mathrm{SOL}(K, F).
\end{aligned}
$$

```
equilibrium
min f('1') x('1') g('1') h('1')
...
min f('N') x('N') g('N') h('N')
vi F pi K
```

An example of using the EMP framework

- An oligopolistic market equilibrium problem formulated as a NEP:

$$q_i^* \in \quad \text{argmax}_{q_i \geq 0} \quad q_i p \left(\sum_{j=1, j \neq i}^{5} q_j^* + q_i \right) - c_i(q_i), \text{ for } i = 1, \ldots, 5.$$

```
variables obj(i); positive variables q(i);
equations defobj(i);
defobj(i)..  obj(i) =E= ...;
model m / defobj /;

file info / '%emp.info%' /;
put info 'equilibrium' /;
loop(i, put 'max', obj(i), q(i), defobj(i) /;);
putclose;
solve m using emp;
```

Special features I: supporting shared constraints

- Shared constraints: agents have shared resources.
- g is a *shared constraint*:

$$\begin{array}{ll} \underset{x_i}{\text{minimize}} & f_i(x_i, x_{-i}), \\ \text{subject to} & g(x_i, x_{-i}) \leq 0. \end{array}$$

- Examples:
  - Network capacity: $\sum_{i=1}^{N} x_i \leq b$
    Agents send packets through the same network channel.
  - Total pollutants: $\sum_{i=1}^{N} a_i x_i \leq c$
    Agents throw pollutants in the river. The maximum pollutants that can be thrown are set by a policy.

Different types of solutions for shared constraints

- A GNEP equilibrium: replicate g and assign a separate multiplier

$$\begin{aligned} \underset{x_i}{\text{minimize}} \quad & f_i(x_i, x_{-i}), \\ \text{subject to} \quad & g(x_i, x_{-i}) \leq 0, \quad (\perp \quad \mu_i \leq 0). \end{aligned}$$

- A variational equilibrium: force use of a single g and a single μ

$$\frac{\underset{x_i}{\text{minimize}} \quad f_i(x_i, x_{-i}) - \mu^T g,}{0 \geq g(x) \quad \perp \quad \mu \leq 0.}$$

- Syntactic enhancement

```
equilibrium
visol g
min f('1') x('1') g
...
min f('N') x('N') g
```

Special features II: supporting shared variables

- Shared variables: agents have shared states.
- y is a *shared variable*:

$$\begin{array}{ll} \underset{y, x_i}{\text{minimize}} & f_i(y, x_i, x_{-i}), \\ \text{subject to} & h(y, x_i, x_{-i}) = 0. \end{array}$$

  - For each x, the value of y is implicitly determined by h.
- Syntactic enhancement

```
equilibrium
implicit y h
min f('1') x('1') y
...
min f('N') x('N') y
```

MCP formulation strategies for shared variables

- Replication

$$F_i(x,y,\mu) = \begin{bmatrix} \nabla_{x_i} f_i - \nabla_{x_i} h \mu_i \\ \nabla_{y_i} f_i - \nabla_{y_i} h \mu_i \\ h \end{bmatrix} \quad \perp \quad \begin{bmatrix} x_i \\ y_i \\ \mu_i \end{bmatrix}$$

- Switching

$$\begin{array}{lcl} F_i(x,y,\mu) = \begin{bmatrix} \nabla_{x_i} f_i - \nabla_{x_i} h \mu_i \\ \nabla_y f_i - \nabla_y h \mu_i \end{bmatrix} & \perp & \begin{bmatrix} x_i \\ \mu_i \end{bmatrix} \\ \hline F_h(x,y,\mu) = \begin{bmatrix} h \end{bmatrix} & \perp & \begin{bmatrix} y \end{bmatrix} \end{array}$$

- Substitution: eliminate $\mu_i \leftarrow [\nabla_y h]^{-1} \nabla_y f_i$

Strategy	Size of the MCP
replication	$(n + 2mN)$
switching	$(n + mN + m)$
substitution (implicit)	$(n + nm + m)$
substitution (explicit)	$(n + m)$

Experimental results: improving sparsity

- Replace $p\left(\sum_{i=1}^{N} x_i\right)$ with $p(y)$ in oligopolistic market problem.
  - 1 ISO agent and 5 energy-producing agents
  - Each energy-producing agent has a fixed number of plants: $n/5$.

n	Original		Switching	
	Size	Density (%)	Size	Density (%)
2,500	2,502	99.92	2,508	0.20
5,000	5,002	99.96	5,008	0.10
10,000	10,002	99.98	10,008	0.05
25,000	-	-	25,008	0.02
50,000	-	-	50,008	0.01

n	Original		Switching	
	(Major,Minor)	Time (secs)	(Major,Minor)	Time (secs)
2,500	(2,2639)	57.78	(1,2630)	1.30
5,000	(2,5368)	420.92	(1,5353)	5.83
10,000	-	-	(1,10517)	22.01
25,000	-	-	(1,26408)	148.08
50,000	-	-	(1,52946)	651.14

Experimental results: modeling mixed behavior

- Revisiting the oligopolistic market equilibrium problem:

$$\underset{q_i \geq 0}{\text{maximize}} \quad q_i p\left(\sum_{j=1, j\neq i}^{5} q_j + q_i\right) - c_i(q_i), \text{ for } i = 1, \ldots, 5.$$

- Introduce a shared variable $y = p(q)$.
  - If an agent declares y as its decision variable, it is a price-maker.
  - Otherwise, it is a price-taker.

Profit	Compet.	Oligo1	Oligo12	Oligo123	Oligo1234	Oligo12345
Firm 1	123.83	125.51	145.59	167.02	185.958	199.93
Firm 2	195.31	216.45	219.63	243.59	264.469	279.72
Firm 3	257.81	278.98	306.17	309.99	331.189	346.59
Firm 4	302.86	322.51	347.48	373.46	376.697	391.28
Firm 5	327.59	344.82	366.54	388.97	408.308	410.36
Total	1207.41	1288.27	1385.42	1483.02	1566.62	1627.875
Soc./wf.	39063.82	39050.19	39034.58	39022.47	39016.37	39015.125

Optimal Value Functions

Problem type

Objective function	or	Constraint
$\min_{x \in X} \theta(x) + \rho(F(x))$		$\min_{x \in X} \theta(x) \text{ s.t. } \rho(F(x)) \leq \alpha$

- Special case is a Quadratic Support Function

$$\rho(y) = \sup_{u \in U} \langle u, By + b \rangle - \frac{1}{2} \langle u, Mu \rangle$$

- Dual representation (of coherent r.m.) in terms of risk sets

$$\rho(Z) = \sup_{\mu \in \mathcal{D}} \mathbb{E}_{\mu}[Z]$$

- If $\mathcal{D} = \{p\}$ then $\rho(Z) = \mathbb{E}[Z]$
- If $\mathcal{D}_{\alpha,p} = \{\lambda \in [0, p/(1-\alpha)] : \langle \mathbb{1}, \lambda \rangle = 1\}$, then $\rho(Z) = \overline{CVaR}_{\alpha}(Z)$

The transformation to MOPEC

- EMP allows any Quadratic Support Function to be defined and facilitates model transformations to tractable forms for solution
- `empinfo` file: OVF cvarup F(x) rho .9

$$\min_{x \in X} \theta(x) + \rho(F(x))$$

$$\rho(y) = \sup_{u \in U} \left\{ \langle u, y \rangle - \frac{1}{2} \langle u, Mu \rangle \right\}$$

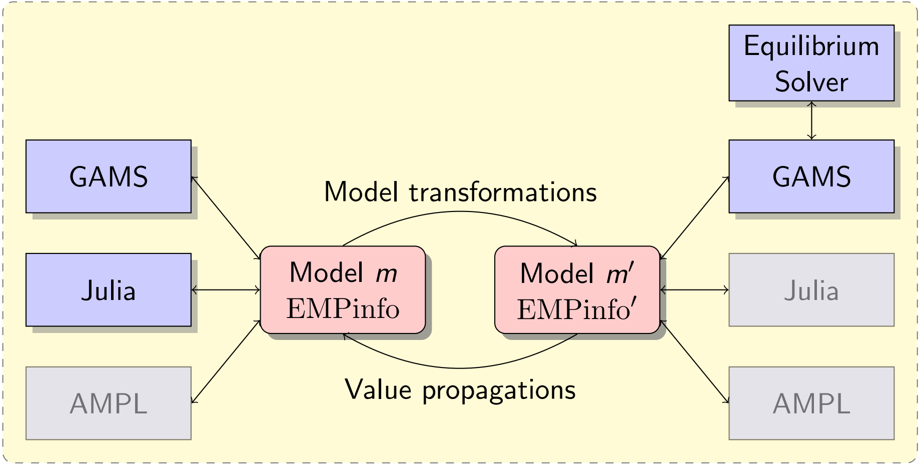
$$0 \in \partial\theta(x) + \nabla F(x)^T \partial\rho(F(x)) + N_X(x)$$

$$0 \in \partial\theta(x) + \nabla F(x)^T u + N_X(x)$$

$$0 \in -u + \partial\rho(F(x)) \iff 0 \in -F(x) + Mu + N_U(u)$$

- This is a MOPEC, and we have a copy of this construct for each agent

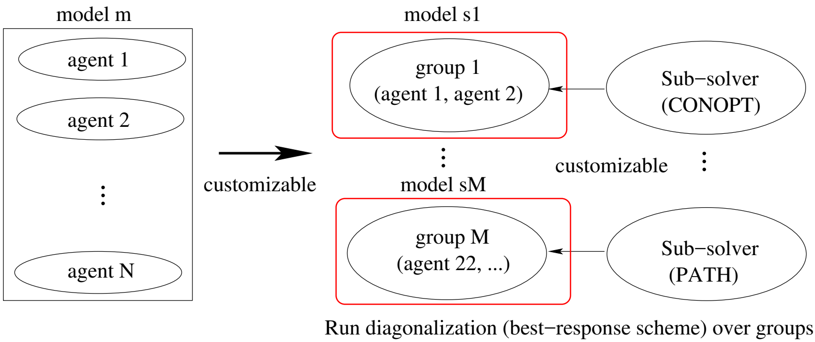
EMP framework



The model representation inside the EMP solver is independent of any model language

Selkie

- Selkie
  - Generates submodels for sub-solvers and decomposition.
  - Supports various decomposition methods.
  - Can compute a solution in an adaptable and flexible way.
  - ex) Selkie on equilibrium problems:



An example of using SELKIE for group diagonalization

- An oligopolistic market equilibrium problem:

$$\underset{q_i \geq 0}{\text{maximize}} \quad q_i p \left(\sum_{j=1, j\neq i}^{5} q_j + q_i \right) - c_i(q_i), \text{ for } i = 1, \ldots, 5.$$

Group	Iterations			
	Jacobi	GS	GSW	GS(RS)
{{1},{2},{3},{4},{5}}	155	45	28	50
{{1,2},{3,4},{5}}	57	21	22	30
{{1..3},{4,5}}	28	14	14	18
{{1..4},{5}}	22	12	12	16
{{1..5}}	1			

  - GS: Gauss-Seidel
  - GSW: Gauss-Southwell
  - GS(RS): Gauss-Seidel with random sweep

- An automatic detection of independent groups is supported.

Multistage MOPEC with risk averse agents

Deterministic equivalent under dual risk measure representation:

$$
\begin{aligned}
\min_{x_{i\cdot},\theta_{i\cdot}} \quad & f_{i0}(x_{i0}) - p_0^T g_{i0}(x_{i0}) + \theta_{i0} \\
\text{s.t.} \quad & \theta_{im} \geq \sum_{n \in m+} \pi_{in}^k \cdot \{ f_{in}(x_{in}) - p_n^T g_{in}(x_{in}) + \theta_{in} \}, \qquad && \forall m \notin L, \ \ k \in K_{im} \\
& x_{in} = H_{in} x_{im} + \omega_{in}, && \forall m \notin L, \ \ n \in m+ \\
& x_{in} \in [l_{in}, u_{in}], && \forall n
\end{aligned}
$$

with equilibrium constraint

$$
0 \leq \sum_a g_{in}(x_{in}) \perp p_n \ \ \geq 0, \quad \forall n
$$

- Here n represents the node of the scenario tree, L is the set of the nodes that are the leaves of the scenario tree, $n+$ represents the set of children nodes of the node n, K_{im} is the set of extreme points of risk set of agent i at node m.

Alternative method: Primal - dual method for solving the multistage MOPEC with risk averse agents

- Previous distributed methods (Gauss-Seidel and Jacobi) fail to solve the MOPEC
  - No implicit function $\pi = h(x)$ from the constraint $0 \leq H(x, \pi) \perp \pi \geq 0$
  - The subproblem is not solvable or unbounded without $H(x, \pi) \geq 0$
- In the risk-averse case, the corresponding reformulated mixed complementarity problem will lose monotonicity even when the risk-neutral case is monotone
- PATH fails to solve, even with informed choices of starting point
- Use Penalty (Augmented Lagrangian) of the constraint $H(x, \pi) \geq 0$ in each primal agents' problem and dual update in each major iterations.
- Performance depends on the choice of γ.

Algorithm 1 Primal-dual for multistage MOPEC with risk-averse agents

set $k = 0$, choose a starting point (x^0, p^0), parameter $\gamma > 0, 0 < \mu \leq 1$
while stopping criterion not met **do**
  for each agent a **do**

$$
\begin{aligned}
x_{i\cdot}^{k+1}, \theta_{i\cdot}^{k+1}, y_{i\cdot}^{k+1} = \arg \min_{x_{i\cdot}, \theta_{i\cdot}, y_{i\cdot}} & \; f_{i0}(x_{i0}) - (p_0^k)^T g_{i0}(x_{i0}) + (p_0^k)^T y_{i0} + \theta_{i0} \\
& + \frac{\gamma}{2} \sum_n \| g_{in}(x_{in}) + \sum_{j<i}^{N} \left(g_{jn}(x_{jn}^{k+1}) - y_j^{k+1} \right) + \sum_{j>i}^{N} \left(g_{jn}(x_{jn}^{k}) - y_j^{k} \right) - b_n \|^2 \\
\text{s.t. } & \; \theta_{im} \geq \sum_{n \in m+} \pi_{in}^k \cdot \{ f_{in}(x_{an}) - p_n^T g_{in}(x_{in}) + p_n^T y_{in} + \theta_{in} \}, \qquad m \notin L, \;\; k \in K_{im} \\
& \; x_{in} = H_{in} x_{im} + \omega_{an}, \quad m \notin L, \;\; n \in m+ \\
& \; x_{in} \in [l_{in}, u_{in}], \;\; y_{in} \geq 0
\end{aligned}
$$

  end for
  $p_n^{k+1} = p_n^k - \mu\gamma \cdot \left(\sum_{j=1}^N g_{jn}(x_{jn}^{k+1}) - b_n - \sum_{j=1}^N y_{jn}^{k+1} \right)$
  $k = k + 1$
end while

Comparison between PATH and Primal-Dual method

- A 4-agent example with 5 stochastic stages, where n represents the dimension of the corresponding MCP
- Risk measure: $\rho(X) = (1-\lambda)\cdot\mathbb{E}(X) + \lambda\cdot CVaR_{0.95}(X)$

Risk averse

Size: $n \times n$	λ	PATH	Primal-Dual		
		Final merit	# Iter	Final merit	Time(secs)
2680 × 2680	0.1	1.15e+03	6887	9.06e-07	7107
2680 × 2680	0.2	1.83e+02	7156	9.35e-05	7200
2680 × 2680	0.3	1.66e+02	7073	3.32e-03	7200
2680 × 2680	0.4	2.71e+02	7083	2.62e-02	7200

Risk averse: use previous solution as initial point

Size: $n \times n$	λ	Primal-Dual		
		# Iter	Final merit	Time(secs)
2680 × 2680	0.1	5179	9.51e-07	4401
2680 × 2680	0.2	5706	9.95e-07	7000
2680 × 2680	0.3	7049	2.52e-06	7200
2680 × 2680	0.4	6967	6.43e-05	7200

Conclusions

- Competition naturally modeled via complementarity
- Solvers exist for medium to large scale problems
- Frameworks (EMP) exist to streamline model transformations
- empinfo: dualvar, bilevel, equilibrium, vi, OVF
- Very large scale models (many agents with many instruments acting strategically) with risk are hard
- Decomposition/diagonalization methods (SELKIE) are effective when sensitivity information is exploited
- New algorithms enable solution of more detailed, authentic problems that address underlying policy questions
- Evaluation via simulation computations and out-of-sample testing